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Caterpillars Have Antimagic Orientations

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Abstract

An *antimagic labeling* of a directed graph D with m arcs is a bijection from the set of arcs of D to $\{1, \dots, m\}$ such that all oriented vertex sums of vertices in D are pairwise distinct, where the *oriented vertex sum* of a vertex u is the sum of labels of all arcs entering u minus the sum of labels of all arcs leaving u. Hefetz, Mütze, and Schwartz [3] conjectured that every connected graph admits an antimagic orientation, where an *antimagic orientation* of a graph G is an orientation of G which has an antimagic labeling. We use a constructive technique to prove that caterpillars, a well-known subclass of trees, have antimagic orientations.

1 Introduction

All graphs considered in this paper are finite and simple unless otherwise stated. A *labeling* of a graph G with m edges is defined as a bijection from the set of edges of G to the set $\{1, \dots, m\}$. A labeling of G is said to be *antimagic* if all vertex sums are pairwise distinct, where the *vertex sum* of a vertex u in G is the sum of labels of all edges incident with u. A graph is said to be *antimagic* if it admits an antimagic labeling. Hartsfield and Ringel conjectured in [4] that all simple connected graphs, with the exception of K_2, are antimagic. Although antimagicness has been proved for graphs belonging to many classes, like regular graphs [1, 2] or trees having more than three vertices and at most one vertex of degree two [10], the conjecture is still open even for bipartite graphs. For related results the reader is referred to the survey of Gallian [5].

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Hartsfield and Ringel's conjecture finds a natural variation in the setting of directed graphs. A *labeling* of a directed graph D with m arcs is a bijection from the set of arcs of D to the set $\{1, \ldots, m\}$. A labeling of D is said to be *antimagic* if all oriented vertex sums are pairwise distinct, where the *oriented vertex sum* of a vertex u in D is the sum of labels of all arcs entering u minus the sum of labels of all arcs leaving u. A graph is said to have an *antimagic orientation* if it has an orientation which admits an antimagic labeling. Hefetz, Mütze, and Schwartz [3] formulate the following conjecture:

Conjecture 1. *Every connected graph admits an antimagic orientation.*

In the same article [3], Hefetz *et al.* prove Conjecture 1 for stars, wheels, cliques, and "dense" graphs (graphs of order n with minimum degree at least $C \log n$, for an absolute constant C); in fact, the authors prove the stronger statement that every orientation is antimagic. Other classes for which Conjecture 1 is known to hold are odd regular graphs [3], even regular graphs [9], or biregular bipartite graphs [11]. Actually, it is easy to see that all antimagic bipartite graphs admit an antimagic orientation where all edges are oriented in the same direction between the partite sets. For this reason, all subclasses of trees that are known to be antimagic admit antimagic orientations. However, a particular subclass for which this reasoning cannot be applied is that of caterpillars.

Definition 1. *A* caterpillar C *is a tree of order at least 3 the removal of whose leaves produces a path.*

When some specific conditions on the number of leaves or on the vertex degrees are added, caterpillars are known to be antimagic [8, 7] but, in the general case, antimagicness is still open for caterpillars. Here we use the flexibility given by the choice of an orientation to adapt the constructive technique from Lozano, Mora, and Seara [8] and prove that all caterpillars admit an antimagic orientation, supporting Conjecture 1.

2 Main Result

Let G be a graph and let u and v be vertices of G. Then, $E(G)$ denotes the set of edges of G and $\{u, v\}$ represents an edge between u and v. An orientation of G will be usually represented by $\vec{G}$, the set of arcs of $\vec{G}$ by $A(\vec{G})$ and an arc from u to v by uv. For any labeling ϕ of a graph G and any orientation $\vec{G}$ of G, we use the notation $\phi(uv)$, for an arc $uv \in A(\vec{G})$, to mean $\phi(\{u, v\})$. This way, ϕ is used to define both the *vertex sum* $s(u)$ of u in G and the *oriented* *vertex sum* $\vec{s}(u)$ of u in $\vec{G}$:

$$s(u) = \sum_{\{u,v\}\in E(G)} \phi(\{u,v\}), \quad \text{and} \quad \vec{s}(u) = \sum_{vu\in A(\vec{G})} \phi(vu) - \sum_{uv\in A(\vec{G})} \phi(uv).$$

For a vertex u in G and a subgraph H of G, $s_H(u)$ represents the vertex sum of u in H. Similarly, for a vertex u in $\vec{G}$ and a subgraph $\vec{H}$ of $\vec{G}$, $\vec{s}_{\vec{H}}(u)$ denotes the oriented vertex sum of u in $\vec{H}$. Given a caterpillar C, Theorem 1 constructs an orientation $\vec{C}$ of C and an antimagic labeling for $\vec{C}$. In the proof, antimagicness will be the consequence of obtaining different absolute values for all oriented vertex sums. For convenience, then, we define the *weight* of a vertex u in an oriented graph $\vec{G}$ as $w(u) = |\vec{s}(u)|$. For a subgraph $\vec{H}$ of $\vec{G}$, $w_{\vec{H}}(u) = |\vec{s}_{\vec{H}}(u)|$ refers to the weight of u in $\vec{H}$.

We also use the notation $[a,b] = \{a, a+1, \ldots, b\}$ for any two integers a, b such that $a \leq b$.

Theorem 1. *Every caterpillar admits an antimagic orientation.*

Proof. Let C be a caterpillar with m edges and r leaves, and consider a longest path $(u_0, \ldots, u_{m-r+2})$ in C. Then, we define the path $P = (u_0, \ldots, u_k)$, where $k = m - r + 2$ if $m - r$ is even and $k = m - r + 1$ otherwise. The number of edges of P is, therefore, even in any case. From here on we will refer to the edges of P as *path edges*, and to the rest of edges in C as *non-path* edges.

We describe now an algorithm in seven steps to construct an orientation $\vec{C}$ of C and a labeling $\phi : A(\vec{C}) \to [1, m]$ that will then be shown to be antimagic.

1. *Defining the label set.* Consider the division of the label set $L = [1, m]$ into the three subsets $L_1 = [1, k_1]$, $L_2 = [k_1+1, k_2]$, and $L_3 = [k_2+1, m]$, where
$$k_1 = \left\lceil \frac{m-r+1}{2} \right\rceil, \quad \text{and} \quad k_2 = \left\lceil \frac{m+r}{2} \right\rceil - 1.$$
Labels in L_1 and L_3 will be used for the path edges, and labels in L_2 for the non-path edges. The equality $k_1 + k_2 = m$ (see Claim 1) will be applied throughout the proof.

2. *Labeling the path edges.* We assign the labels in L_1 and L_3 to the path edges in an alternating way. The largest label in L_1, k_1, is assigned to the edge $\{u_0, u_1\}$, then the largest label in L_3, m, is assigned to the next edge, $\{u_1, u_2\}$, then the previous labels are assigned to the next edges in P and so on, keeping the alternation of labels until the first ones are reached. The sequence of labels assigned to the path edges is depicted in Figure 1. Formally, for $1 \leq i < k$, the labeling is defined as:

$$\phi(\{u_i, u_{i+1}\}) = \begin{cases} k_1 - \frac{i}{2}, & \text{if } i \text{ is even} \\ m - \frac{i-1}{2}, & \text{otherwise} \end{cases}$$

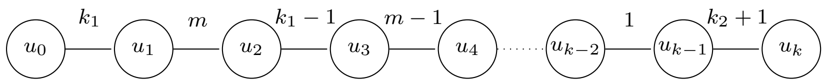

Figure 1: Labeling of the path P.

3. *Classifying the path vertices and the non-path edges.* Let u_i be a path vertex with degree at least two in C. Then, we call u_i *light* if $s_P(u_i) < m$ and u_i is adjacent to exactly one vertex not belonging to P; otherwise, u_i is called *heavy.* Note that the path vertices of degree one, in particular u_0, and also u_k when $m - r$ is even, have degree one in C and, then, are neither light nor heavy.

   A non-path edge is called *light* if it is incident with a light vertex, and *heavy* if it is incident with a heavy vertex.

4. *Orienting the path edges.* The edge $\{u_0, u_1\}$ is oriented as the arc u_0u_1. The edge $\{u_i, u_{i+1}\}$, for $0 < i < k$, is oriented in the "same direction" in the path as the edge $\{u_{i-1}, u_i\}$ if u_i is light and in "contrary direction" if u_i is heavy, that is, as:

   (a) u_iu_{i+1} if either $u_{i-1}u_i \in A(\vec{C})$ and u_i is light or $u_iu_{i-1} \in A(\vec{C})$ and u_i is heavy,

   (b) $u_{i+1}u_i$ if either $u_{i-1}u_i \in A(\vec{C})$ and u_i is heavy or $u_iu_{i-1} \in A(\vec{C})$ and u_i is light.

   The oriented path P will be represented with $\vec{P}$.

5. *Orienting the non-path edges.* Note that any non-path edge is incident with a light or a heavy vertex. We consider two cases for a non-path edge $\{u_i, v\}$ where u_i is a path vertex and v is a leaf:

   (a) if u_i is light, we orient $\{u_i, v\}$ as u_iv if $\vec{s}_{\vec{P}}(u_i) > 0$, and as vu_i if $\vec{s}_{\vec{P}}(u_i) < 0$,

   (b) if u_i is heavy, we orient $\{u_i, v\}$ as u_iv if $\vec{s}_{\vec{P}}(u_i) < 0$, and as vu_i if $\vec{s}_{\vec{P}}(u_i) > 0$.

   The goal of this orientation is that heavy edges help maximizing the weight of the heavy vertices they are incident with, while light edges help minimizing the weight of their respective light vertices.

6. *Labeling the light edges.* Let $u_{i_1}, u_{i_2}, \dots, u_{i_{n_l}}$ be an ordering of the n_l light vertices such that $w_{\vec{P}}(u_{i_1}) \le w_{\vec{P}}(u_{i_2}) \le \cdots \le w_{\vec{P}}(u_{i_{n_l}})$. Let l_t be the light edge incident with u_{i_t}, for $1 \le t \le n_l$; then, we define $\phi(l_t) = k_2 - t + 1$.

7. *Labeling the heavy edges.* For each heavy vertex u_j with at least two incident heavy edges, we randomly assign unused labels from L_2 to all heavy edges incident with u_j except one. Now, all heavy vertices must be incident with at most one heavy edge which has not yet been assigned a label. Let $u_{j_1}, u_{j_2}, \dots, u_{j_{n_h}}$ be an ordering of the n_h heavy vertices having one incident still unlabeled heavy edge such that $w'(u_{j_1}) \le w'(u_{j_2}) \le \cdots \le w'(u_{j_{n_h}})$, where $w'(u_{j_1}), w'(u_{j_2}), \dots, w'(u_{j_{n_h}})$ are the partial weights calculated with the labels assigned so far. Let h_t be the still unlabeled heavy edge incident with j_t, for $1 \le t \le n_h$; then, we define $\phi(h_t)$ as the t-th smallest unused label in L_2.

Let $\vec{C}$ be the orientation of C defined above. Now, we establish some facts before proving that ϕ is an antimagic labeling of $\vec{C}$.

Claim 1. *It holds that $k_1 + k_2 = m$.*

Proof. We use well-known transformations of the ceiling and floor functions [6], as indicated in the side annotations for a real x and an integer n:

$$\begin{aligned}
m &= m + r - r \\
&= \left\lfloor \frac{m+r}{2} \right\rfloor + \left\lceil \frac{m+r}{2} \right\rceil - r && \text{since } n = \lfloor n/2 \rfloor + \lceil n/2 \rceil \\
&= \left\lfloor \frac{m-r}{2} \right\rfloor + \left\lceil \frac{m+r}{2} \right\rceil && \text{since } \lfloor x \rfloor + n = \lfloor x + n \rfloor \\
&= \left\lceil \frac{m-r-1}{2} \right\rceil + \left\lceil \frac{m+r}{2} \right\rceil && \text{since } \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n-1}{2} \right\rceil \\
&= \left\lceil \frac{m-r+1}{2} \right\rceil + \left\lceil \frac{m+r}{2} \right\rceil - 1 && \text{since } \lceil x \rceil + n = \lceil x + n \rceil
\end{aligned}$$

□

Claim 2. *For any t such that $1 \le t \le n_l$, it holds that $k_2 \le w_{\vec{P}}(u_{i_t}) \le k_2 + 1$.*

Proof. Let u_{i_t} be a light vertex. If $i_t = k = m - r + 1$ (in the case $m - r$ is odd), then we know that u_{i_t} is an endpoint of the path and is incident with a light edge; therefore, $w_{\vec{P}}(u_{i_t}) = k_2 + 1$ from step 2 in the algorithm. Otherwise, since u_0 is not a light vertex, suppose u_{i_t} is not an endpoint of the path. The orientation of the edges done in step 4 implies that the weight of

u_{i_t} in the path, $w_{\vec{P}}(u_{i_t})$, is the result of subtracting the labels of the edges incident with u_{i_t} (defined in step 2):

$$w_{\vec{P}}(u_{i_t}) = |\phi(\{u_{i_t-1}, u_{i_t}\}) - \phi(\{u_{i_t}, u_{i_t+1}\})|.$$

If i_t is even, then

$$\begin{aligned} w_{\vec{P}}(u_{i_t}) = \left|\left(m - \frac{(i_t-1)-1}{2}\right) - \left(k_1 - \frac{i_t}{2}\right)\right| &= (m - i_t/2 + 1) - (k_1 - i_t/2) \\ &= m - k_1 + 1 = k_2 + 1, \end{aligned}$$

where Claim 1 has been applied at the last equality. Similarly, if i_t is odd, then

$$w_{\vec{P}}(u_{i_t}) = \left|\left(k_1 - \frac{i_t-1}{2}\right) - \left(m - \frac{i_t-1}{2}\right)\right| = |k_1 - m| = |-k_2| = k_2.$$

Therefore, $k_2 \leq w_{\vec{P}}(u_{i_t}) \leq k_2 + 1$ for any light vertex u_{i_t}. □

Claim 3. *It holds that $n_l \leq k_1 - 1$.*

Proof. For any light vertex u_i it holds that $s_P(u_i) < m$. Since all possible vertex sums in the path that are smaller than m are $k_2+1, k_2+2, \ldots, k_2+k_1-1 = m-1$, the maximum number of light vertices n_l must necessarily be at most $k_1 - 1$. □

Now we prove that ϕ is an antimagic labeling of $\vec{C}$. We do this by classifying the vertices into different classes depending on their degree and on the fact of being light or heavy. Then, we show that the weights are pairwise distinct inside each class and that the sets of weights for the classes are pairwise disjoint. As a conclusion, any oriented vertex sums at two different vertices in $\vec{C}$ must be distinct because, otherwise, their absolute values (that is, their weights) would coincide.

Light vertices

Note that, as a consequence of the orientation (step 5) and labeling (step 6) of the light edges, the weight of a light vertex u_{i_t}, for $1 \leq t \leq n_l$, is $w(u_{i_t}) = |w_{\vec{P}}(u_{i_t}) - (k_2 - t + 1)|$.

In the first place, we observe that the weights of any two light vertices u_{i_s} and u_{i_t}, with $1 \leq s < t \leq n_l$, must be different:

$$\begin{aligned} w(u_{i_s}) &= |w_{\vec{P}}(u_{i_s}) - (k_2 - s + 1)| \\ &= w_{\vec{P}}(u_{i_s}) - (k_2 - s + 1) && \text{since } w_{\vec{P}}(u_{i_s}) \geq k_2 \text{ by Claim 2} \\ &\leq w_{\vec{P}}(u_{i_t}) - (k_2 - s + 1) && \text{since } w_{\vec{P}}(u_{i_s}) \leq w_{\vec{P}}(u_{i_t}) \text{ by assumption} \\ &= |w_{\vec{P}}(u_{i_t}) - (k_2 - s + 1)| && \text{since } w_{\vec{P}}(u_{i_t}) \geq k_2 \text{ by Claim 2} \\ &< |w_{\vec{P}}(u_{i_t}) - (k_2 - t + 1)| \\ &= w(u_{i_t}). \end{aligned}$$

In the second place, we show that the weights of light vertices are distinct from the weights of the rest of vertices in C. The smallest weight of a light vertex is that of u_{i_1}, $w(u_{i_1}) = |w_{\vec{P}}(u_{i_1}) - k_2|$, which is 0 or 1 by Claim 2. As for the largest weight of a light vertex, we have

$$\begin{aligned} w(u_{i_{n_l}}) &= |w_{\vec{P}}(u_{i_{n_l}}) - (k_2 - n_l + 1)| \\ &\leq |(k_2 + 1) - (k_2 - n_l + 1)| && \text{by Claim 2} \\ &\leq |(k_2 + 1) - (k_2 - (k_1 - 1) + 1)| && \text{by Claim 3} \\ &= k_1 - 1. \end{aligned}$$

Therefore, the weights of light vertices are pairwise distinct and belong to the set $[0, k_1 - 1]$.

Vertices of degree one

Vertices of degree one that belong to the path are at most two: u_0, with weight k_1, and u_k if $m - r$ is even, in which case $w(u_k) = k_2 + 1$. Vertices of degree one not belonging to the path, however, have a weight corresponding to the value of a label in L_2, hence belonging to the set $[k_1 + 1, k_2]$. Clearly, then, all weights of vertices of degree one are pairwise distinct and belong to the set $[k_1, k_2 + 1]$, being larger than the weights of light vertices.

Heavy vertices with no incident heavy edge

Let u_j be a heavy vertex with no incident heavy edge. Since heavy vertices have degree at least two in C, u_j must have two path vertices as neighbors and, then, its incident path edges have been oriented in contrary directions in step 4, that is, either as $u_{j-1}u_j$ and $u_{j+1}u_j$ or as u_ju_{j-1} and u_ju_{j+1}. Then, the labeling ϕ defined in step 2 (see also Figure 1) ensures that if u_j and $u_{j'}$ are two heavy vertices with no incident heavy edge, for $0 < j < j' < k$, we have $w(u_j) > w(u_{j'})$, being $m + k_1$ the largest weight and $k_2 + 2$ the smallest one.

Therefore, the weights of all heavy vertices with no incident heavy edge are pairwise distinct and belong to $[k_2+2, m+k_1]$, being larger than the weights of light vertices and of vertices of degree one.

Heavy vertices with incident heavy edges

The orientation of a heavy edge defined in step 5 ensures that any label assigned to it contributes to an increase in the weight of the heavy vertex it is incident with. Consider the list of nondecreasing partial weights $w'(u_{j_1}) \leq w'(u_{j_2}) \leq \cdots \leq w'(u_{j_{n_h}})$ from step 7, and consider two partial weights $w'(u_{j_s})$ and $w'(u_{j_t})$ from this list, with $1 \leq s < t \leq n_h$; remember also that h_s and h_t are the unlabeled heavy edges incident with, respectively, u_{j_s} and u_{j_t}. Then, we have that

$$w(u_{j_s}) = w'(u_{j_s}) + \phi(h_s) < w'(u_{j_s}) + \phi(h_t) \leq w'(u_{j_t}) + \phi(h_t) = w(u_{j_t})$$

and, therefore, the weights of any two different heavy vertices u_{j_s} and u_{j_t} are always different.

Moreover, we can argue that the weight of any heavy vertex u_j with incident heavy edges is $w(u_j) \geq m + k_1 + 1$. We consider the two possibilities for a heavy vertex u_j according to its definition:

1. $w_{\vec{P}}(u_j) < m$ *and* u_j *has at least two incident heavy edges.* In this case, $w_{\vec{P}}(u_j) \geq k_2 + 1$ and the sum of the weights from its incident heavy edges must be at least $2k_1 + 3$, the sum of the two smallest labels in L_2. Then, $w(u_j) \geq (k_2 + k_1) + k_1 + 4 = m + k_1 + 4$.

2. $w_{\vec{P}}(u_j) \geq m$ *and* u_j *has at least one incident heavy edge.* In this case, the sum of the weights from its incident heavy edges is at least $k_1 + 1$, the value of the smallest label in L_2. Then, $w(u_j) \geq m + k_1 + 1$.

Therefore, in any case we have that $w(u_j) \geq m + k_1 + 1$, which is larger than the weight of any other vertex in the previous classes. □

As an example of Theorem 1, consider the caterpillar depicted in Figure 2. In the notation of the theorem, it has $m = 16$ edges and $r = 10$ leaves. Since a longest path has 8 edges, which is even, we define P as $(u_0, \ldots, u_8)$ without shrinking the longest path by one. In step 1 we set the values $k_1 = 4$, and $k_2 = 12$ and divide the label set $[1, 16]$ into $L_1 = [1, 4]$, $L_2 = [5, 12]$, and $L_3 = [13, 16]$. Then, P is labeled alternating the labels from L_1 and L_3, as indicated in step 2. Now, step 3 identifies vertex u_6 as light because $s(u_6) = 15 < m$ (its incident path edges are then oriented in the same direction in step 4), while the rest of path vertices which are not endpoints are defined

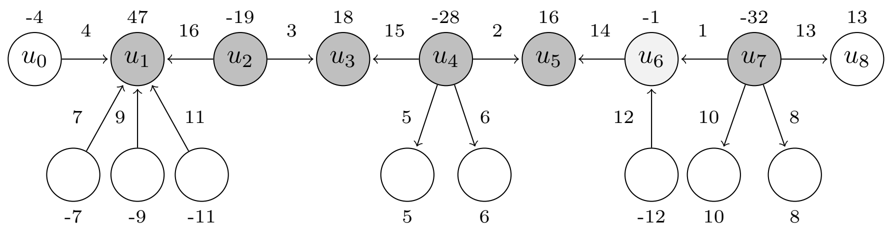

Figure 2: Antimagic orientation of a caterpillar. The only light vertex is u_6, colored in light grey; heavy vertices are colored in dark grey.

as heavy vertices (and their incident path edges oriented in contrary directions in step 4). In step 5, heavy edges are oriented in the direction that maximizes the weight of their respective heavy vertices, while the only light edge (incident with u_6) is oriented in such a way that the weight of u_6 be minimized. This is now done in step 6, where label 12, the largest one in L_2, is assigned to the light edge incident with u_6. Step 7 starts assigning random labels from L_2 to four heavy edges, labels 7, 9, 5, and 10 in the example, leaving one heavy edge unlabeled for each of the vertices u_1, u_4, and u_7. The way to assign the remaining labels to the still unlabeled heavy edges is the following. First, calculate the list of partial weights of the heavy vertices having incident heavy edges by nondecreasing weight: $w'(u_4) = 22$, $w'(u_7) = 24$, and $w'(u_1) = 36$. Then, assign the remaining labels by increasing value (6, 8, and 11) to the unlabeled heavy edge incident with each vertex in the list. The final weights are strictly increasing in the same ordering and, therefore, pairwise distinct.

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